

Objectives:

- Review the chain rule for derivatives.
- Define and use u-substitution for integrals.

Review: Compute the following derivatives:

1. $h(x) = f(g(x))$

$$h'(x) = f'(g(x))g'(x)$$

2. $f(x) = (x^3 + 2x)^5$

$$f'(x) = 5(x^3 + 2x)^4(3x + 2)$$

3. $g(x) = e^{x^2}$

$$g'(x) = e^{x^2}(2x)$$

4. $h(t) = (\sin(t))^3$

$$h'(t) = 3(\sin(t))^2 \cos(t)$$

5. $k(x) = \ln(\tan(x))$

$$k'(x) = \frac{1}{\tan(x)} (\sec(x))^2$$

6. $\ell(w) = \arctan(\cos(w))$

$$\ell'(w) = \frac{1}{1 + (\cos(x))^2} (-\sin(x))$$

Substitution:

So, we now know $\int \frac{(\sec(x))^2}{\tan(x)} dx = \frac{\ln(\tan(x)) + C}{}$.

In general, $\int f'(g(x))g'(x) dx = \frac{f(g(x)) + C}{}$.

Another way we can write this is $\int f'(u) du = \frac{f(u) + C}{}$. We call this substituting u for $g(x)$ and often refer to this method as “u-substitution”.

Examples:

- $\int e^{\sin(x)} \cos(x) dx$

Pick $u = \sin(x)$ so then $du = \cos(x)dx$.

$$\int e^u du = e^u + C = e^{\sin(x)} + C$$

We can check by taking the derivative: $\frac{d}{dx}e^{\sin(x)} = e^{\sin(x)} \cos(x)$

- $\int 3x^2 (x^3 + 5)^{10} dx$

Pick $u = x^3 + 5$ so then $du = 3x^2 dx$.

$$\int u^{10} du = \frac{u^{11}}{11} + C = \frac{(x^3 + 5)^{11}}{11} + C$$

(Notice that not every choice for u is equally helpful. For example, if we picked $u = x^3$ we would've been stuck with the integral $\int (u + 5)^{10} du$, which is not as nice as $\int u^{10} du$ at all. In general, we might have to do some experimentation to see which choice of u [if any] simplifies the integral the best.)

- $\int \frac{1}{t+2} dt$

$u = t + 2, du = (1)dt$

$$\int \frac{1}{u} du = \ln(u) + C = \ln(t + 2) + C$$

Sometimes we have to manipulate the integral before using substitution:

Example: $\int x^2 (x^3 + 7)^5 dx$

Method 1: Solve for dx :

If we want $u = x^3 + 7$ then $du = 3x^2 dx$ so $\frac{1}{3x^2} du = dx$. So,

$$\int x^2 (x^3 + 7)^5 dx = \int x^2 u^5 \left(\frac{1}{3x^2} du \right) = \int \frac{1}{3} u^5 du = \frac{1}{3} \cdot \frac{u^6}{6} = \frac{(x^3 + 7)^6}{18}$$

Method 2: “Fix-it-up”:

We want $u = x^3 + 7$ so we’d really like to see $3x^2 dx$ in our integral. Massage the integral so it looks like this:

$$\int x^2 (x^3 + 7)^5 dx = \int \frac{1}{3} \cdot 3x^2 (x^3 + 7)^5 dx = \int \frac{1}{3} u^5 du$$

Then proceed in the same way as in method 1.

Definite Integrals Using u-Substitution:

Example: $\int_0^1 x^2 (1 + 2x^3)^4 dx$

Method 1 : Change limits of integration:

We pick $u = 1 + 2x^3$, $du = 6x^2$. Then if $x = 0$, $u = 1 + 2(0)^3 = 1$ and if $x = 1$, $u = 1 + 2(1)^3 = 3$. So we have:

$$\int_0^1 x^2 (1 + 2x^3)^4 dx = \int_1^3 \frac{1}{6} u^4 du = \frac{1}{6} \cdot \frac{u^5}{5} \Big|_1^3 = \frac{1}{6} \cdot \frac{3^5}{5} - \frac{1}{6} \cdot \frac{1^5}{5} = \frac{243}{30} - \frac{1}{30} = \frac{242}{30}$$

Method 2 : Find integral in terms of original variable, then substitute.

Use same u , du as above but keep the x limits of integration:

$$\int_0^1 x^2 (1 + 2x^3)^4 dx = \int_{x=0}^{x=1} \frac{1}{6} u^4 du = \frac{1}{6} \cdot \frac{u^5}{5} \Big|_{x=0}^{x=1} = \frac{1}{6} \cdot \frac{(1 + 2x^3)^5}{5} \Big|_0^1 = \frac{(1 + 2(1)^3)^5}{30} - \frac{(1 + 2(0)^3)^5}{30} = \frac{242}{30}$$